# MATH 5061 Problem Set 3[1]

**Due date:** Mar 11, 2024

**Problems:** (Please hand in your assignments by submitting your PDF via email. **Late submissions will not be accepted.**)

Throughout this assignment, we use $(M, g)$ to denote a smooth $n$-dimensional Riemannian manifold with its Levi-Civita connection $\nabla$ unless otherwise stated. The Riemann curvature tensor (as a $(0,4)$-tensor) of $(M, g)$ is denoted by $R$.

1. Suppose that $(M^n, g)$ is a connected Riemannian manifold with $n \geq 3$ such that there exists a function $f : M \to \mathbb{R}$ such that $K(\sigma) = f(p)$ for all two-dimensional subspace $\sigma \subset T_pM$. Show that $f$ must be a constant function on $M$.(*Hint: use the second Bianchi identity*)

2. A Riemannian manifold $(M^n, g)$ is called *Einstein manifold* if there exists a smooth function $\lambda : M \to \mathbb{R}$ such that $\text{Ric}(X, Y) = \lambda\langle X, Y\rangle$ for any vector fields $X, Y \in \Gamma(TM)$.
   (a) Suppose $(M^n, g)$ is a connected Einstein manifold with $n \geq 3$, show that $\lambda$ must be a constant function.
   (b) Suppose $(M^3, g)$ is a connected 3-dimensional Einstein manifold. Show that $M$ has constant sectional curvature.

3. Let $f : M \to \mathbb{R}$ be a smooth function defined on a Riemannian manifold $(M^n, g)$. Denote $\Sigma := f^{-1}(a)$ where $a$ is a regular value of $f$. Show that the mean curvature $H$, with respect to the unit normal $N = -\frac{\nabla f}{|\nabla f|}$, of the hypersurface $\Sigma$ is given by $H = \pm \text{div} N$ (up to a sign depending on the sign convention in the definition of mean curvature).

4. Consider the smooth map $F : \mathbb{R}^2 \to \mathbb{R}^4$ defined by
$$F(u, v) = (\cos u, \sin u, \cos v, \sin v).$$
   (a) Show that $F$ is an isometric immersion (with respect to the flat metrics).
   (b) Prove that the image of $F$ lies inside the round 3-sphere $\mathbb{S}^3 := \{x \in \mathbb{R}^4 \mid |x|^2 = 2\}$, and $\Sigma = F(\mathbb{R}^2)$ is a minimal immersion into $\mathbb{S}^3$, equipped with the induced metric from $\mathbb{R}^4$.

[1] Last revised on February 25, 2024